Introduction to Longitudinal Structural Equation Modeling: Theory
Topics we will cover

● Track I
  ○ Raw versus residualized change
  ○ Path Diagrams
  ○ Autoregressive Cross-Lag Panel Models + extensions
  ○ Latent Curve Model
    ■ Fixed and random effects

● Track II
  ○ Latent Curve Models review + extensions
  ○ Multivariate LCMs + extensions
  ○ Latent Change Score Models
    ■ Univariate
    ■ Multivariate

We have data:

But which model do we pick??
Introduction to Longitudinal Structural Equation Modeling: Theory
Track I
Different Kinds of Change over Time

Consider $y_1$ and $y_2$ as repeated measures

1. Residualized Change
   - Differences in residuals (conditioning on prior information)

2. Raw Score Change
   - Difference scores
   - Smooth trajectories
   - Latent change scores
     - A special case of both approaches
Different Kinds of Change over Time

Residualized Change

- Change in value of $y_2$ beyond what is expected based on $y_1$
  - Can use a regression expression to predict $y_2$ from $y_1$
  - $y_{2i} = \beta_0 + \beta_1 y_{1i} + \varepsilon_i$
  - $\hat{y}_{2i} = \beta_0 + \beta_1 y_{1i}$
  - $y_{2i} = \hat{y}_{2i} + \varepsilon_i$

- So residualized change is $\varepsilon_i = y_{2i} - \hat{y}_{2i}$
  - i.e., change from the model’s expectation

- Used in ANCOVA and ARCL models
Different Kinds of Change over Time

Residualized Change

● Pros
  ○ $y_1$ and $y_2$ do not even have to be the same measure (remember it’s just a regression)
    ■ May be important in contexts of measurement non-invariance (a fun term that we won’t have time to go into, but bother Ethan or John about it sometime if you suffer from insomnia (why do we love what hurts us?))
  ○ Can help control for background characteristic differences in some contexts

● Cons
  ○ Inconsistent with most longitudinal theories which concern absolute change (i.e., trajectories)
  ○ Regressions link pairs of timepoints at a time rather than the full timeseries
    ■ Parameter rich models
Different Kinds of Change over Time

Raw Score Change

- Quite literally based on the difference in magnitude between $y_2$ and $y_1$
  - $d_i = y_{2i} - y_{1i}$

- Forms the basis of most longitudinal models
  - Paired t-test (which is barely a longitudinal model), repeated measures ANOVA, growth models, latent change score models
    - “two waves of data are better than one, but maybe not much better” (Rogosa et al., 1982)
  - For most of these models we don’t directly compute difference scores, but change on the raw metric is the unit of analysis
Different Kinds of Change over Time

Raw Score Change

● Pros
  ○ Intuitive and what you would likely think of in terms of longitudinal modeling

● Cons
  ○ If there is measurement error on \( y_1 \) and/or \( y_2 \) (and there will be; heartbreak #1 of the day), that will get compounded in the difference score
  ○ Unreliability will decrease the power of our model test of change (heartbreak #2)
  ○ **Fair warning:** we’ll generally accept these limitations for reasons (we’ll explain when you’re older; in approx 20 minutes)
Different Kinds of Change over Time

- Of course, people have calm, rational disagreements about this
  - Just kidding, quantitative methodologists are **dramatic**, especially in the 20th century

- Cronbach & Furby (1970); (yes that Cronbach)
  - “...they [raw change scores] are still employed, even by some otherwise sophisticated investigators.”

- Willett (1997)
  - “When discussing residualized change scores, methodologists disagree as to what exactly is being estimated, how well it is estimated, and how it can be interpreted.”
  - So that's all...

- Lord’s paradox (as solved by Judea Pearl, 2016)
Different Kinds of Change over Time: Readings


Equations and Path Diagrams in SEM

- SEM can be entirely expressed in terms of equations of course, but a handy visual shortcut has been developed to represent models that is often more intuitive for users
  - The important thing to realize though is that the visuals are isomorphic with the underlying equations, meaning that a full diagram implies all the same information as the equations (although there are some shortcuts we’ll take to reduce visual clutter)

- We’ll talk about 2 broad types of path diagrams that we’ll see variations of
  - Path models (e.g., observed mediation analysis, autoregressive cross-lag models)
  - Factor models (e.g., growth models being a special case)
Equations and Path Diagrams in SEM

- If we consider a simple 2-predictor regression, the equation is as follows:
  - $y_i = \beta_0 + \beta_1x_{1i} + \beta_2x_{2i} + \varepsilon_i$ with a normally distributed error term, $\varepsilon_i \sim N(0, \sigma^2)$
  - In SEM, we generally use slightly different notations
    - Intercepts: $\alpha$
    - Regression coefficients of $y$'s on $x$'s: $\gamma$
    - Regression coefficients of $y$'s on other $y$'s: $\beta$
    - Residuals: $\zeta$ with variance $\psi$
  - So now we express the above as: $y_i = \alpha + \gamma_1x_{1i} + \gamma_2x_{2i} + \zeta_i$ where $\zeta_i \sim N(0, \psi)$
Equations and Path Diagrams in SEM

- However we can represent \( y_i = \alpha + \gamma_1 x_{1i} + \gamma_2 x_{2i} + \zeta_i \) where \( \zeta_i \sim N(0, \Psi) \) visually too.

- **Diagram conventions**
  - Observed variables are rectangles
  - Latent variables (including residuals, remember we don’t directly observe them) are circles
  - Single-headed arrows are regression coefficients and double-headed arrows are covariances
Equations and Path Diagrams in SEM

- Of course path models are usually more complicated than simple regression and allow for multiple dependent (y) variables
  - A simple case below:

\[
\begin{align*}
  y_{1i} &= \alpha_1 + \gamma_{11}x_{1i} + \gamma_{12}x_{2i} + \zeta_{1i} \\
  y_{2i} &= \alpha_2 + \gamma_{21}x_{1i} + \beta_{21}y_{1i} + \zeta_{2i}
\end{align*}
\]

Model Equations:

\[
\begin{align*}
  y_{1i} &= \alpha_1 + \gamma_{11}x_{1i} + \gamma_{12}x_{2i} + \zeta_{1i} \\
  y_{2i} &= \alpha_2 + \gamma_{21}x_{1i} + \beta_{21}y_{1i} + \zeta_{2i}
\end{align*}
\]
Very often we drop alphas, psi’s and gammas/betas and imply their existence.

This would be to display your theoretical model; you could also put model estimates in place of the parameters in the results.
Equations and Factor Diagrams in SEM

A factor in SEM represents a latent variable.

A latent variable is some quantity that we *infer* from a set of observations.

Examples you’ve seen in this workshop and in the wild:

- Literally any computed score from a questionnaire
- The slope and intercept in a MLM growth model
  - In MLM, we’re focus on the population mean of these (i.e., the fixed effects)
  - But we get the variance of the latent variables as the SD of the random effects

We’ll cover the SEM growth model later, but it’s really a restricted (i.e., constrained) form of a factor model.
Equations and Factor Diagrams in SEM

- The equation (bold lowercase terms are vectors; bold uppercase are matrices)
  \[ y_i = \Lambda \eta_i + \epsilon_i \]

- Remember that we do not observe \( \eta \), but can infer its existence if we assume this model
Equations and Factor Diagrams in SEM

The scale score example

- When you take the average of a set of items and call that “Internalizing”, you’re creating a sort of latent variable model with some very strong constraints.
  - It’s a formative rather than reflective model
- Every item is equally formative of the latent variable.
- Every item and the construct is measured without any error.
Equations and Factor Diagrams in SEM

Latent variables (in SEM, MLM, and other contexts) allow us to model the true score of a variable free from measurement error.

This is awesome.

Again, feel free to bug us about measurement anytime.
Equations and Diagrams: Readings

We saw before that **residualized change** is just a form of regression of a given measure on itself at a previous time.

- With SEM’s ability to have multiple dependent variables, we can string several repeated measures together using a path model.

Note that we simplify the path model further by having orphaned arrows indicate the residuals.
Autoregressive Cross-Lag Panel Models: Theory

- Of course this is somewhat boring with a single variable
  - Raw change scores are often univariate (i.e., using a change score as the outcome of another variable), but ARCLs are almost never
  - Can test how two variables “travel together” over time
    - autoregressive and cross-lagged effects

One of the ONLY times quantitative methods people name something exactly what it is. Of course they also call these cross-lag panel models
Autoregressive Cross-Lag Panel Models: Path Diagram

- Autoregressive Effects estimate the stability of a construct across time
  - Estimated separately for each construct (orange for y’s, green for z’s)
  - And for each lag, unless you impose equality constraints
Autoregressive Cross-Lag Panel Models: Path Diagram

- Cross-Lag Effects estimate the prospective prediction of $y$ on $z$ (orange), net the effect of $y$ at the previous time point
  - And vice versa for $z$ on $y$ (green)
- Within time measures (or their residuals) are allowed to correlate (blue)
Autoregressive Cross-Lag Panel Models: Expansions

● Can include
  ○ additional repeated measures and look at chains of regression across time
  ○ exogenous predictors
  ○ Higher order lagged regression paths (e.g., lag-2)

● Instead of observations at each time point, could model a latent variable
  ○ VERY parameter intensive

● Can impose equality constraints to reduce parameter overload (a technical term)
Autoregressive Cross-Lag Panel Models: Limitations & Extensions

- Without equality constraints, ARCLs are highly parameterized models
  - Difficult to interpret change over time
  - No best practices for ordering of equality constraints

- Does not model change in means at all (can’t get a trajectory out of these models)

- Fails to disaggregate within-person change from between person differences
  - There is a fix (known as RI-ARCL [aka RI-CLPM] or more generally, structured residual growth models)
  - See more in Track II

- Probably don’t ever use a standard ARCL model. Please.
Random Intercept ARCL model

- Never use the standard ARCL model (aka CLPM).
- Use this model, the RI-ARCL (RI-CLPM) which separates out between and within person variance.
- You can easily specify and fit this model with the **riclpm package** (handles more than 2 vars!)
- Here’s a more in-depth tutorial: [https://johnflournoy.science/2017/10/20/riclpm-lavaan-demo/](https://johnflournoy.science/2017/10/20/riclpm-lavaan-demo/)
Autoregressive Cross-Lag Panel Models: Readings


Latent Curve Model: Theory

- Latent variables are all around (Bollen, 2002)
  - Constructs that we do not (or cannot) directly measure but believe to exist
    - Actually, we cannot directly measure anything
      - Do you remember high-school physics, measuring everything 3 times?
    - Depression, cognitive control, etc

- In the context of growth, we assume that people start somewhere and grow in a certain direction
  - BUT, we can’t actually measure true growth. We use repeated observations to infer the trajectory
  - If you’ve ever fit a random slope mixed effect growth model, you’ve already used latent variables!
  - Let’s look at some pictures

Seriously, read everything this man writes about SEM
Latent Curve Model: Theory

- We have repeated measures, but how can we assess change?
Latent Curve Model: Theory

● Could connect each time point
  ○ This would assess change interval by interval (a collection of difference scores)
  ○ This assumes that each observation is measured without error, and that change is linear between observations. Weird, right?
Latent Curve Model: Theory

- OR, could fit a smooth line to the data
  - Creates a trajectory with a model-implied intercept and slope
  - Separates true growth from measurement error
  - Rests on the assumption of linearity (first degree)
Latent Curve Model: Theory

- Trajectories attempt to capture underlying development in repeated measures
  - Views repeated measures as composed of both true change (a smooth underlying trajectory) and noise (time specific deviation/residuals of the observed scores from the trajectory)

- For a broad class of basic models, is mathematically identical to the mixed effect growth model
  - Some good reading: Curran, 2003; Bauer, 2003
  - SEM can be expanded to much more complex models that no longer have mixed effect equivalents (see Track II for some examples)
Latent Curve Model: Path Diagram

● To build this trajectory, we use a highly constrained confirmatory factor model
  ○ For a first-degree linear model, we define two factors (1 intercept and 1 linear slope)
  ○ Identity of the factor is determined by setting factor loadings to specific values

● Wait, what’s it mean to identify a factor? (very briefly)
  ○ SEM is a system of multiple equations.
  ○ The number of unknown parameters must be less than the number of pieces of information you give the model. Setting loadings to specific values leaves fewer parameters unknown (to be estimated)
Latent Curve Model: Path Diagram

**Measurement Equation**

\[ y_i = \Lambda \eta_i + \epsilon_i \]

**Structural Equation**

\[ \eta_i = \alpha + \zeta_i \]

Model Implied Mean Structure

\[ \mu(\theta) = \Lambda \alpha \]

Model Implied Covariance Structure

\[ \Sigma(\theta) = \Lambda \Psi \Lambda' + \Theta \]

This part is basically magic. We’re trying to reproduce the means and covariance of the observed variables with only the model structure....
Latent Curve Model: It’s very similar to MLM

This is essentially:

\[ y_{ij} = \gamma_{00} + \gamma_{10} \text{TIME} + \nu_{00} + \nu_{00} + \epsilon_{ij}, \]

where TIME is 0, 1, 2, 3 (structured appropriately for the observations)

Notice that TIME is centered at the first timepoint implicitly in this model (see track 2 for more).
Latent Curve Model: Fixed vs Random Effects

- Just like in mixed-effect growth models we can have fixed (population mean) and random (individual) trajectories
  - We either set the variance of the factor to zero (fixed only) or freely estimate (fixed + random)
  - We never estimate individual trajectories, but we can compute model-implied trajectories on the back end

- Probably easier to see visually
Latent Curve Model: Fixed vs Random Effects

Fixed Intercept, Fixed Slope

Group and individual trajectories are stacked on top of one another

You would never do this, but it's basically OLS regression

Random Intercept, Fixed Slope

Group (thick) and individual (thin) trajectories only differ in level (parallel lines)
Latent Curve Model: Fixed vs Random Effects

Fixed Intercept, Random Slope

- Non-parallel lines radiate from a common intercept
- Relatively uncommon but sometimes makes sense

Random Intercept, Random Slope

- Individuals differ both in intercept and rate of change over time
- Notice that rank order is NOT preserved across time
Latent Curve Model: Extensions

- Can vary the number/nature of factors
  - Intercept only, linear, quadratic, splines, latent basis

- Can bring in predictors and outcomes at the structural and measurement levels

- Multivariate Models
  - The **real** unique strength of SEM
  - but see also Bayesian modeling packages like brms and languages like Stan, too

- Structuring the residuals
  - Separation of within and between person variance
  - Combines the strengths of the ARCL without many of the limitations
Latent Curve Model: Readings


Introduction to Longitudinal Structural Equation Modeling: Theory Track II
Latent Curve Model: A quick review

Measurement Equation

\[ y_i = \Lambda \eta_i + \varepsilon_i \]

Structural Equation

\[ \eta_i = \alpha + \zeta_i \]

Model Implied Mean Structure

\[ \mu(\theta) = \Lambda \alpha \]

Model Implied Covariance Structure

\[ \Sigma(\theta) = \Lambda \Psi \Lambda' + \Theta \]

This part is basically magic. We're reproducing the means and covariance of the observed variables with only the factor structure… (okay so we may be a bit lame [really me mainly])
Latent Curve Model: A quick review

- **Centering**
  - Where we code time as zero influences the interpretation of the intercept and covariance of the latent factors, but model fit will be identical
  - In higher order models, centering will affect all but the highest-order factor

- **Time structure**
  - Time is coded directly into the factor loadings and in traditional LCMs we need observed variables at discrete time points
  - Missing data is easily accommodated but we need “sufficient” observations at each discrete time point
  - Continuous time can be included with definition variables/tscore model (need other software)
Latent Curve Model: Readings


Latent Curve Model: Nonlinear Trajectories

● Higher-order Polynomials
  ○ A quadratic trend can be imposed by squaring the linear factor loadings
  ○ Often very difficult to estimate a quadratic random effect
    ■ That’s fine, can just fix the variance to zero
  ○ Even higher polynomials can be specified by this is almost never done
Latent Curve Model: Nonlinear Trajectories

- **Piecewise Splines**
  - One alternative to a smooth polynomial is to do local curve approximation with discontinuous functions.
  - For linear piecewise models, you need at least 5 timepoints (3 for each line w/ 1 shared “knot point”).
  - Advanced applications can have non-linear pieces and can even estimate the knot point.
Latent Curve Model: Nonlinear Trajectories
Latent Curve Model: Nonlinear Trajectories

- **Latent Basis Models**
  - When we think developmental patterns may be complex non-linear, we might fit a free-loading model
    - Allows a subset of factor loadings to be freely estimated
    - Can fit a complex developmental surface
  - Comes with some limitations
    - Can no longer interpret beta as a per-unit change in y
    - Random effects of “the shape”?  
    - Lots of potential to overfit the data
Latent Curve Model: Nonlinear Trajectories

\[ \psi_{11} \xi_1 \xi_2 \psi_{12} \]

\[ \lambda_{11} \lambda_{12} \lambda_{22} \]

\[ \alpha_1 \alpha_2 \]

\[ \eta_1 \eta_2 \]

\[ \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4 \]

\[ y_1 y_2 y_3 y_4 \]
Nonlinear Latent Curve Model: Readings


LCM: Time-Invariant Covariates (TICs)

- Predictors that vary at the person level, but do not take on unique values over time
  - Theoretical: age at first child birth, childhood adversity, treatment group
  - Convenience: brain function at age 10, self-identified sex

- Enter the model at the latent variable level
  - TICs can predict conditional changes in the intercept and slope
  - Just a quick note: intercept and slope contain no temporal information
LCM: Time-Invariant Covariates

**Measurement Equation**

\[ y_i = \Lambda \eta_i + \varepsilon_i \]

**Structural Equation**

\[ \eta_i = \alpha + \Gamma x_i + \zeta_i \]
LCM: Time-Varying Covariates (TVCs)

- Predictors that take on unique values over time
  - Truly time-specific effects: weather, treatment dosage
  - Contain person-level information too: almost **everything** we do (e.g., depression on stress)

- Enter the model at the manifest/observed variable level
  - TVCs predict time-specific deviations from the underlying trajectory
    - Can include contemporaneous and lagged effects
  - But also conditions trajectory on the TVC so inclusion can change the implied trajectory
    - E.g., from quadratic to linear

- TVCs should **not** be systematically changing over time
LCM: Time-Varying Covariates

**Measurement Equation**

$$y_i = \Lambda\eta_i + Kz_i + \varepsilon_i$$

**Structural Equation**

$$\eta_i = \alpha + \zeta_i$$
Latent Curve Model with Covariates: Readings


Multivariate LCM

- What if we think our TVC is systematically changing over time?
  - Or just contains person-level information
  - We can model a growth process on y and z at the same time

- A raw score change approach to modeling how two or more variables travel together over time
  - Co-development: Hancock & Curran, 2021
Multivariate LCM

Measurement Equation
\[ y_i \mid z_i = \Lambda \eta_i + \varepsilon_i \]

Structural Equation
\[ \eta_i = \alpha + \zeta_i \]

\[ \Psi = \begin{bmatrix} \psi_{11} & \psi_{21} & \psi_{22} \\ \psi_{31} & \psi_{32} & \psi_{33} \end{bmatrix} \]

\[ \Theta = \begin{bmatrix} \theta_{y_1} & 0 & 0 & \theta_{y_2} \\ 0 & \theta_{y_2} & 0 & 0 \\ 0 & 0 & \theta_{y_3} & 0 \\ \theta_{z_1} & 0 & 0 & \theta_{z_1} \\ \theta_{z_1} & \theta_{z_2} & 0 & 0 \\ \theta_{z_2} & 0 & \theta_{z_3} & 0 \\ \theta_{z_3} & 0 & 0 & \theta_{z_4} \\ \theta_{z_4} & 0 & 0 & 0 \end{bmatrix} \]
Multivariate LCM: Extensions

- Can bring in TICs to predict growth factors
  - Could do cross-construct regression among factors, but I recommend against

- Can have different function forms for each outcome
  - Example was intercept only for z, intercept + linear slope for y
  - Could have higher polynomials

- Can separate within- and between-person effects by structuring residuals
Within- vs. Between-Person Effects

- Most theories in DCN posit within-person processes
  - However often evaluated with between-person statistics
  - Even in longitudinal models (Curran & Bauer, 2011)

- TVCs that systematically vary in level confound within- and between-person effects

- Well developed procedures in mixed-effects models, but relatively rare to see in SEMs (which too be clear, is a bad thing)
Residual Structure

Between-person variance is isolated to the latent factor structure of the model.

Within-person variance is isolated to the structured residual regressions.
Multivariate Latent Curve Model: Readings


Latent Change Score Model: 2 timepoints

- Can re-express a raw change score as a special case of the autoregressive model where we set the autoregressive parameter to 1
  - Allows us to solve for the residual in the model: $\zeta_i = y_{2i} - (1)\cdot y_{1i} - \alpha_i = d_i - \alpha_i$
  - Normally don’t have access to the residual in the model but using this approach, we can create a phantom variable that we can then use in the model
LCS Model: 2 timepoints

- To put raw change in the model, we need to put an additional latent factor on top of the phantom variables
  - This is not really latent in the way we normally think

- This model is an SEM expression of a paired samples t-test
  - Only difference is it allows missing data
  - Can build on this simple model to do more interesting things
LCS Model: Univariate Latent Change

- The means of the delta latent variables can tell us something about nature of change
  - Close to zero: no change
  - Equal means: constant change
  - Unequal means: discontinuous change

- Can use these factors as predictors of other variables of interest
  - Highly flexible but can also be hard to theoretically interpret
LCS Model: Proportional Change

- We can regress latent change ($\eta_{\Delta(t-1)}$) on prior level ($\eta_{yt}$)
  - Implies that the amount of change that occurs is in-part determined by prior level
- Typically $\beta$ is set equal across time but not necessary
- Actually identical to the ARCL model with AR(1) lags
  - These initial LCS models are really just re-parameterizing models we are more familiar with
LCS Model: Trajectories

- **Spoiler alert:** we can also re-formulate the LCM as an LCS
  - But we'll see unique LCS features after this

- **Basically just push y into phantom variables**
  - Phantoms don’t add model complexity
  - One more reformulation before we can get to the good stuff
LCS Model: Trajectories

- We can also use the $\eta_\Delta$ formulation
  - Intercept is now defined on only a single phantom variable
  - But note that the slope loadings are now all 1
    - Essentially sums across discrete change (which is what a smooth trajectory should do)
- This is the formulation that allows for something we haven’t seen before
LCS Model: Trajectories

- Adding the proportional change parameter is a unique way to model non-linearities
  - Especially good for exponential trends
  - But cannot fit random effects here

- An interesting enough model, but the really unique case is when we move to the bivariate model with proportional change
Multivariate Dual-Change Model

- Here we model non-linear changes **within** each construct separately

- We get cross-construct correlations at the latent level like the multivariate LCM
Multivariate Dual-Change Model

- But can use cross-construct regressions to look at proportional linkage across variables

- Incredibly flexible and powerful model
  - What theory does this model describe/test?
Latent Change Score Model: Readings


Where do we go from here?